Multiple Choice Questions

1. If $a = 8^{53}$, $b = 16^{41}$ and $c = 64^{27}$, then which of the following inequalities is true?

   (A) $a > b > c$ (B) $c > b > a$ (C) $b > a > c$ (D) $b > c > a$ (E) $c > a > b$

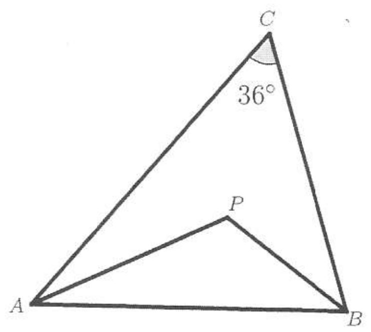
2. If a, b, c are real numbers such that $|a-b| = 1$, $|b-c| = 1$, $|c-a| = 2$ and $abc = 60$, find the value of $\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c}$.

   (A) $\frac{1}{30}$ (B) $\frac{1}{20}$ (C) $\frac{1}{10}$ (D) $\frac{1}{4}$ (E) None of the above

3. If x is a complex number satisfying $x^2 + x + 1 = 0$, what is the value of $x^{49} + x^{50} + x^{51} + x^{52} + x^{53}$?

   (A) -1 (B) $-\frac{1}{2}$ (C) 0 (D) $\frac{1}{2}$ (E) 1

4. In $\triangle ABC$, $\angle ACB = 36^\circ$ and the interior angle bisectors of $\angle CAB$ and $\angle ABC$ intersect at P. Find $\angle APB$.



   (A) 72° (B) 108° (C) 126° (D) 136° (E) None of the above

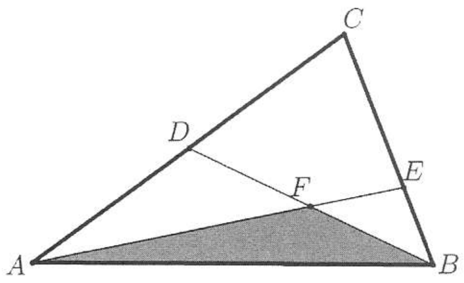
5. Find the number of integer pairs x, y such that $xy - 3x + 5y = 0$.

   (A) 1 (B) 2 (C) 4 (D) 8 (E) 16

6. Five young ladies were seated around a circular table. Miss Ong was sitting between Miss Lim and Miss Mak. Ellie was sitting between Cindy and Miss Nai. Miss Lim was between Ellie and Amy. Lastly, Beatrice was seated with Miss Poh on her left and Miss Mak on her right. What is Daisy's surname?

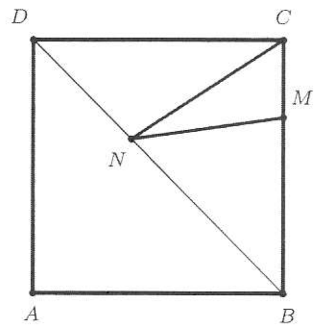
   (A) Lim (B) Mak (C) Nai (D) Ong (E) Poh

7. Given that ABC is a triangle with D being the midpoint of AC and E a point on CB such that $CE = 2EB$. If AE and BD intersect at point F and the area of $\triangle AFB = 1$ unit, find the area of $\triangle ABC$.



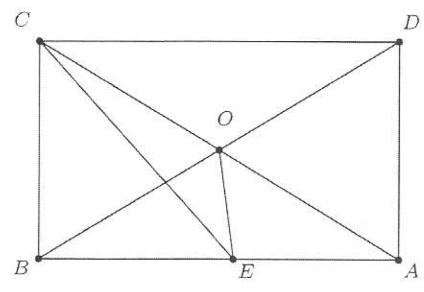
(A) 3 (B) $\frac{10}{3}$ (C) $\frac{11}{3}$ (D) 4 (E) 5

8. $ABCD$ is a square with sides 8 cm. M is a point on CB such that $CM = 2$ cm. If N is a variable point on the diagonal DB, find the least value of $CN + MN$.



(A) 8 (B) $6\sqrt{2}$ (C) 10 (D) $8\sqrt{2}$ (E) 12

9. $ABCD$ is a rectangle whose diagonals intersect at point O. E is a point on AB such that CE bisects $\angle BCD$. If $\angle ACE = 15°$, find $\angle BOE$.



(A) 60° (B) 65° (C) 70° (D) 75° (E) 80°

10. Let S be the smallest positive multiple of 15, that comprises exactly $3k$ digits with k '0's, k '3's and k '8's. Find the remainder when S is divided by 11.

(A) 0 (B) 3 (C) 5 (D) 6 (E) 8

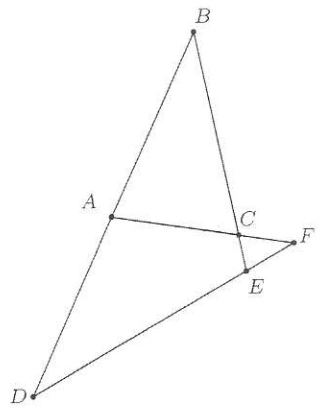
Short Questions

11. Find the value of $\sqrt{9999^2 + 19999}$.

12. If the graphs of $y = x^2 + 2ax + 6b$ and $y = x^2 + 2bx + 6a$ intersect at only one point in the xy-plane, what is the x-coordinate of the point of intersection?

13. Find the number of multiples of 11 in the sequence $99, 100, 101, 102, \cdots, 20130$.

14. In the figure below, BAD, BCE, ACF and DEF are straight lines. It is given that $BA = BC$, $AD = AF$, $EB = ED$. If $\angle BED = x°$, find the value of x.

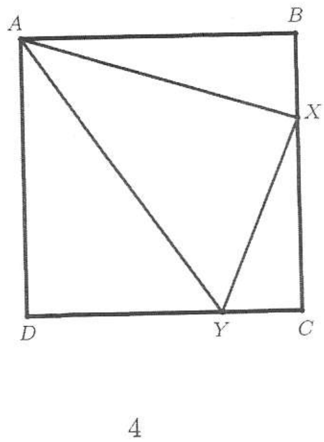


15. If $a = 1.69$, $b = 1.73$ and $c = 0.48$, find the value of

$$\frac{1}{a^2 - ac - ab + bc} + \frac{2}{b^2 - ab - bc + ac} + \frac{1}{c^2 - ac - bc + ab}.$$

16. Suppose that x_1 and x_2 are the two roots of the equation $(x-2)^2 = 3(x+5)$. What is the value of the expression $x_1 x_2 + x_1^2 + x_2^2$?

17. Let $ABCD$ be a square and X and Y be points such that the lengths of XY, AX and AY are 6, 8 and 10 respectively. The area of $ABCD$ can be expressed as $\frac{m}{n}$ units where m and n are positive integers without common factors. Find the value of $m + n$.



18. Let x and y be real numbers satisfying the inequality

$$5x^2 + y^2 - 4xy + 24 \leq 10x - 1.$$

Find the value of $x^2 + y^2$.

19. A painting job can be completed by Team A alone in 2.5 hours or by Team B alone in 75 minutes. On one occasion, after Team A had completed a fraction $\frac{m}{n}$ of the job, Team B took over immediately. The whole painting job was completed in 1.5 hours. If m and n are positive integers with no common factors, find the value of $m + n$.

20. Let a, b and c be real numbers such that $\dfrac{ab}{a+b} = \dfrac{1}{3}$, $\dfrac{bc}{b+c} = \dfrac{1}{4}$ and $\dfrac{ca}{c+a} = \dfrac{1}{5}$. Find the value of $\dfrac{24abc}{ab + bc + ca}$.

21. Let x_1 and x_2 be two real numbers that satisfy $x_1 x_2 = 2013$. What is the minimum value of $(x_1 + x_2)^2$?

22. Find the value of $\sqrt{45 - \sqrt{2000}} + \sqrt{45 + \sqrt{2000}}$.

23. Find the smallest positive integer k such that $(k - 10)^{4026} \geq 2013^{2013}$.

24. Let a and b be two real numbers. If the equation $ax + (b - 3) = (5a - 1)x + 3b$ has more than one solution, what is the value of $100a + 4b$?

25. Let $S = \{1, 2, 3, \ldots, 48, 49\}$. What is the maximum value of n such that it is possible to select n numbers from S and arrange them in a circle in such a way that the product of any two adjacent numbers in the circle is less than 100?

26. Given any 4-digit positive integer x not ending in '0', we can reverse the digits to obtain another 4-digit integer y. For example if x is 1234 then y is 4321. How many possible 4-digit integers x are there if $y - x = 3177$?

27. Find the least positive integer n such that $2^8 + 2^{11} + 2^n$ is a perfect square.

28. How many 4-digit positive multiples of 4 can be formed from the digits 0, 1, 2, 3, 4, 5, 6 such that each digit appears without repetition?

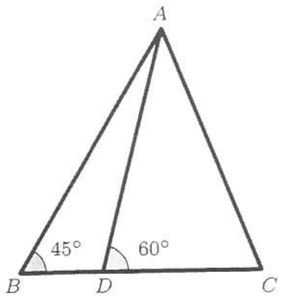
29. Let m and n be two positive integers that satisfy

$$\frac{m}{n} = \frac{1}{10 \times 12} + \frac{1}{12 \times 14} + \frac{1}{14 \times 16} + \cdots + \frac{1}{2012 \times 2014}.$$

Find the smallest possible value of $m + n$.

30. Find the units digit of $2013^1 + 2013^2 + 2013^3 + \cdots + 2013^{2013}$.

31. In $\triangle ABC$, $DC = 2BD$, $\angle ABC = 45°$ and $\angle ADC = 60°$. Find $\angle ACB$ in degrees.



32. If a and b are positive integers such that $a^2 + 2ab - 3b^2 - 41 = 0$, find the value of $a^2 + b^2$.

33. Evaluate the following sum

$$\left\lfloor \frac{1}{1} \right\rfloor + \left\lfloor \frac{1}{2} \right\rfloor + \left\lfloor \frac{2}{2} \right\rfloor + \left\lfloor \frac{1}{3} \right\rfloor + \left\lfloor \frac{2}{3} \right\rfloor + \left\lfloor \frac{3}{3} \right\rfloor + \left\lfloor \frac{1}{4} \right\rfloor + \left\lfloor \frac{2}{4} \right\rfloor + \left\lfloor \frac{3}{4} \right\rfloor + \left\lfloor \frac{4}{4} \right\rfloor + \left\lfloor \frac{1}{5} \right\rfloor + \cdots,$$

up to the 2013^{th} term.

34. What is the smallest possible integer value of n such that the following statement is always true?

In any group of $2n - 10$ *persons, there are always at least* 10 *persons who have the same birthdays.*

(For this question, you may assume that there are exactly 365 different possible birthdays.)

35. What is the smallest positive integer n, where $n \neq 11$, such that the highest common factor of $n - 11$ and $3n + 20$ is greater than 1?

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2013

Junior Section (First Round Solutions)

Multiple Choice Questions

1. Answer: (D)

   First note that $c = (8^2)^{27} = 8^{54}$, so we see that $c > a$. Next, $b = (4^2)^{41} = 4^{82}$ and $c = (4^3)^{27} = 4^{81}$. Therefore we have $b > c$. Consequently $b > c > a$.

2. Answer: (B)

$$\begin{aligned}\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}-\frac{1}{a}-\frac{1}{b}-\frac{1}{c} &= \frac{a^2+b^2+c^2-bc-ca-ab}{abc}\\ &= \frac{(a^2-2ab+b^2)+(b^2-2bc+c^2)+(c^2-2ca+a^2)}{2abc}\\ &= \frac{(a-b)^2+(b-c)^2+(c-a)^2}{2abc}\\ &= \frac{1^2+1^2+2^2}{120} = \frac{1}{20}.\end{aligned}$$

3. Answer: (A)

   Note that $x^2+x+1 = \frac{x^3-1}{x-1}$, so $x^2+x+1=0$ implies that $x^3=1$ and $x \neq 1$. Now

$$\begin{aligned}x^{49}+x^{50}+x^{51}+x^{52}+x^{53} &= x^{49}(1+x+x^2)+x^{51}(x+x^2)\\ &= x^{49}\times 0+(x^3)^{17}(-1)\\ &= 1^{17}\times(-1) = -1.\end{aligned}$$

4. Answer: (B)

   Let $\angle CAB = x$ and $\angle ABC = y$. Then $x+y = 180^\circ - 36^\circ = 144^\circ$.
   Now $\angle APB = 180^\circ - \frac{x+y}{2} = 108^\circ$.

5. Answer: (D)

   $xy - 3x + 5y = 0$ is equivalent to $(x+5)(y-3) = -15$.
   If $x+5 = a$ and $y-3 = b$, then there are eight distinct pairs of integers a, b (counting signs) such that $ab = -15$.

6. Answer: (B)

   Beatrice, being between Miss Poh and Miss Mak cannot be Miss Ong who was between Miss Lim and Miss Mak. This means that we have in order from the left, Miss Poh, Beatrice, Miss Mak, Miss Ong and Miss Lim. So Beatrice must be Miss Nai. Since Ellie was beside Miss Nai and also besides Miss Lim, she must be Miss Poh. This implies Cindy is Miss Lim and Amy was Miss Ong leaving Daisy as Miss Mak.

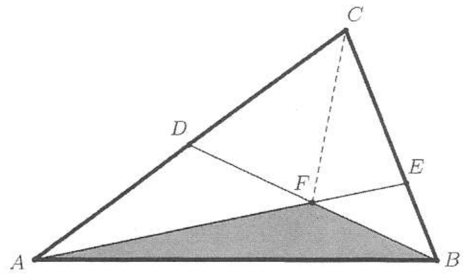
7. Answer: (D)

   Construct a line joining C and F. Then using $[XYZ]$ to denote the area of $\triangle XYZ$, we know that $[ADF] = [DCF] = x$ and if $[BFE] = z$, then $[FCE] = 2z$.

   Furthermore, we have $[ADB] = [DCB]$ i.e. $x + 1 = x + 3z$, so $z = \frac{1}{3}$.

   Also, $2 \times [AEB] = [ACE]$ i.e. $2 + 2z = 2x + 2z$, so $x = 1$.

   In conclusion, $[ABC] = 1 + 2x + 3z = 4$ units.



8. Answer: (C)

   Join A to N. By symmetry, $AN + NM = MN + CN$, and the least value occurs when ANM is a straight line. Thus the least value is

$$\sqrt{AB^2 + BM^2} = \sqrt{8^2 + 6^2} = 10.$$

9. Answer: (D)

   Since CE bisects $\angle BCD$, $\angle BCE = 45°$. Thus $\angle CEB = 45°$ also and $\triangle CBE$ is isosceles. Therefore $BC = BE$.

   Now $\angle BCO = 45° + 15° = 60°$. As $CO = BO$, we conclude that $\triangle COB$ is equilateral. Thus $BC = BO = BE$ giving us an isosceles triangle OBE. Since $\angle OBE = 30°$, thus $\angle BOE = 75°$.

10. Answer: (D)

    S being a multiple of 5 and 3 must end with '0' and has the sum of digits divisible by 3. Since $3 + 8 = 11$, the smallest positive k such that $k \times 11$ is divisible by 3 is 3. Thus $S = 300338880$ and the remainder is

$$0 - 8 + 8 - 8 + 3 - 3 + 0 - 0 + 3 = -5 \equiv 6 \pmod{11}.$$

Short Questions

11. Answer: 10000

$\sqrt{9999^2+19999}=\sqrt{9999^2+2\times 9999+1}=\sqrt{(9999+1)^2}=10000.$

12. Answer: 3

Let (α,β) be the point of intersection of the two graphs. Then

$$\beta=\alpha^2+2a\alpha+6b=\alpha^2+2b\alpha+6a.$$

It follows that $2(a-b)\alpha=6(a-b)$. Since the two graphs intersect at only one point, we see that $a-b\neq 0$ (otherwise the two graphs coincide and would have infinitely many points of intersection). Consequently $2\alpha=6$, and hence $\alpha=3$.

13. Answer: 1822

The the number of multiples of 11 in the sequence $1,2,...,n$ is equal to $\lfloor \frac{n}{11}\rfloor$. Thus the answer to this question is $\left\lfloor \frac{20130}{11}\right\rfloor-\left\lfloor\frac{98}{11}\right\rfloor=1830-8=1822.$

14. Answer: 108

Let $\angle ABC=\alpha$ and $\angle BAC=\beta$. Since $BA=BC$, we have $\angle BCA=\angle BAC=\beta$. As $EB=ED$, it follows that $\angle EDB=\angle EBD=\angle ABC=\alpha$. Then $\angle AFD=\angle ADF=\angle EDB=\alpha$ since $AD=AF$. Note that $\angle DAF=180^\circ-\beta$. In $\triangle ABC$, we have $\alpha+2\beta=180^\circ$; and in $\triangle ADF$, we have $2\alpha+180^\circ-\beta=180^\circ$. From the two equations, we obtain $\alpha=36^\circ$. By considering $\triangle BDE$, we obtain $x=180^\circ-2\alpha=108^\circ$.

15. Answer: 20

$$\begin{aligned}
&\frac{1}{a^2-ac-ab+bc}+\frac{2}{b^2-ab-bc+ac}+\frac{1}{c^2-ac-bc+ab}\\
&=\frac{1}{(a-b)(a-c)}+\frac{2}{(b-a)(b-c)}+\frac{1}{(c-a)(c-b)}\\
&=\frac{c-b-2(c-a)-(a-b)}{(a-b)(b-c)(c-a)}\\
&=\frac{-1}{(a-b)(b-c)}=20.
\end{aligned}$$

16. Answer: 60

The equation $(x-2)^2=3(x+5)$ is equivalent to $x^2-7x-11=0$. Thus $x_1+x_2=7$ and $x_1x_2=-11$. So

$$x_1x_2+x_1^2+x_2^2=(x_1+x_2)^2-x_1x_2=7^2-(-11)=60.$$

17. Answer: 1041

Let the length of the side be s. Observe that since $6^2 + 8^2 = 10^2$ so $\angle AXY = 90°$. This allows us to see that $\triangle ABX$ is similar to $\triangle XCY$. Thus $\frac{AX}{XY} = \frac{AB}{XC}$, i.e. $\frac{8}{6} = \frac{s}{s - BX}$. Solving this equation gives $s = 4BX$ and we can then compute that

$$8^2 = AB^2 + BX^2 = 16BX^2 + BX^2.$$

So $BX = \frac{8}{\sqrt{17}}$ and $s^2 = 16 \times \frac{64}{17} = \frac{1024}{17}$. Thus $m + n = 1041$.

18. Answer: 125

The inequality is equivalent to

$$(x-5)^2 + (2x-y)^2 \leq 0.$$

Thus we must have $(x-5) = 0$ and $(2x - y) = 0$, hence $x^2 + y^2 = 5^2 + 10^2 = 125$.

19. Answer: 6

Suppose Team B spent t minutes on the job. Then

$$\frac{t}{75} + \frac{90-t}{150} = 1.$$

Thus $t = 60$ minutes and so Team A completed $\frac{30}{150} = \frac{1}{5}$ of the job. So $m + n = 6$.

20. Answer: 4

Taking reciprocals, we find that $\frac{1}{a} + \frac{1}{b} = 3$, $\frac{1}{b} + \frac{1}{c} = 4$ and $\frac{1}{a} + \frac{1}{c} = 5$. Summing the three equations, we get

$$12 = 2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 2 \times \frac{ab + bc + ca}{abc}.$$

Hence $\frac{24abc}{ab + bc + ca} = 4$.

21. Answer: 8052

$$(x_1 + x_2)^2 = (x_1 - x_2)^2 + 4x_1x_2 \geq 0 + 4 \times 2013 = 8052.$$

If $x_1 = x_2 = \sqrt{2013}$, then $(x_1 + x_2)^2 = 8052$.

22. Answer: 10

Let $x_1 = \sqrt{45 - \sqrt{2000}}$ and $x_2 = \sqrt{45 + \sqrt{2000}}$. Then $x_1^2 + x_2^2 = 90$ and

$$x_1x_2 = \sqrt{(45 - \sqrt{2000})(45 + \sqrt{2000})} = \sqrt{45^2 - 2000} = \sqrt{25} = 5.$$

Thus

$$(x_1 + x_2)^2 = x_1^2 + x_2^2 + 2x_1x_2 = 100.$$

As both x_1 and x_2 are positive, we have $x_1 + x_2 = 10$.

23. Answer: 55

$(k-10)^{4026} = ((k-10)^2)^{2013} \geq 2013^{2013}$ is equivalent to to $(k-10)^2 \geq 2013$. As $k-10$ is an integer and $44^2 < 2013 < 45^2$, the minimum value of $k-10$ is 45, and thus the minimum value of k is 55.

24. Answer: 19

Rearranging the terms of the equation, we obtain

$$(1-4a)x = 2b+3.$$

Since the equation has more than one solution (i.e., infinitely many solutions), we must have $1-4a=0$ and $2b+3=0$. Therefore $a=\dfrac{1}{4}$ and $b=-\dfrac{3}{2}$. Consequently, $100a+4b=19$.

25. Answer: 18

First note that the product of any two different 2-digit numbers is greater than 100. Thus if a 2-digit number is chosen, then the two numbers adjacent to it in the circle must be single-digit numbers. Note that at most nine single-digit numbers can be chosen from S, and no matter how these nine numbers $1,2,\ldots,9$ are arranged in the circle, there is at most one 2-digit number in between them. Hence it follows that $n \leq 18$. Now the following arrangement

$$1,49,2,33,3,24,4,19,5,16,6,14,7,12,8,11,9,10,1$$

shows that $n \geq 18$. Consequently we conclude that the maximum value of n is 18.

26. Answer: 48

Let $x=\overline{abcd}$ and $y=\overline{dcba}$ where $a,d \neq 0$. Then

$$\begin{aligned} y-x &= 1000\times d - d + 100\times c - 10\times c + 10\times b - 100\times b + a - 1000\times a \\ &= 999(d-a) + 90(c-b) = 9\left(111(d-a)+10(c-b)\right). \end{aligned}$$

So we have $111(d-a)+10(c-b)=353$. Consider the remainder modulo 10, we obtain $d-a=3$, which implies that $c-b=2$. Thus the values of a and b determines the values of d and c respectively.

a can take on any value from 1 to 6, and b can take any value from 0 to 7, giving $6\times 8=48$ choices.

27. Answer: 12

Let $2^8+2^{11}+2^n=m^2$ and so

$$2^n = m^2 - 2^8(1+8) = (m-48)(m+48).$$

If we let $2^k=m+48$, then $2^{n-k}=m-48$ and we have

$$2^k - 2^{n-k} = 2^{n-k}(2^{2k-n}-1) = 96 = 2^5\times 3.$$

This means that $n-k=5$ and $2k-n=2$, giving us $n=12$.

28. Answer: 208

Note that a positive integer k is a multiple of 4 if and only if the number formed by the last two digits of k (in the same order) is a multiple of 4. There are 12 possible multiples of 4 that can be formed from the digits $0, 1, 2, 3, 4, 5, 6$ without repetition, namely

$$20, 40, 60, 12, 32, 52, 04, 24, 64, 16, 36, 56.$$

If 0 appears in the last two digits, there are 5 choices for the first digit and 4 choices for the second digit. But if 0 does not appear, there are 4 choices for the first digit and also 4 choices for the second digit. Total number is

$$4 \times 5 \times 4 + 8 \times 4 \times 4 = 208.$$

29. Answer: 10571

$$\begin{aligned}\frac{m}{n} = \frac{1}{4}\sum_{k=5}^{1006} \frac{1}{k(k+1)} &= \frac{1}{4}\sum_{k=5}^{1006} \frac{1}{k} - \frac{1}{k+1} \\ &= \frac{1}{4}\left(\frac{1}{5} - \frac{1}{1007}\right) \\ &= \frac{501}{10070}.\end{aligned}$$

Since $\gcd(501, 10070) = 1$, we have $m + n = 10571$.

30. Answer: 3

Note that the units digit of $2013^1 + 2013^2 + 2013^3 + \cdots + 2013^{2013}$ is equal to the units digit of the following number

$$3^1 + 3^2 + 3^3 + \cdots + 3^{2013}.$$

Since $3^2 = 9, 3^3 = 27, 3^4 = 81$, the units digits of the sequence of $3^1, 3^2, 3^3, 3^4, \cdots, 3^{2013}$ are

$$\underbrace{3, 9, 7, 1, 3, 9, 7, 1, \cdots, 3, 9, 7, 1}_{\text{2012 numbers}}, 3.$$

Furthermore the sum $3 + 9 + 7 + 1$ does not contribute to the units digit, so the answer is 3.

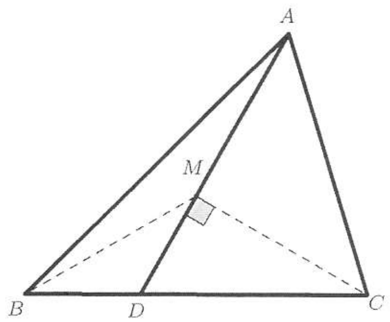
31. Answer: 75

Construct a point M on AD so that CM is perpendicular to AD. Join B and M.

Since $\angle ADC = 60°$, $\angle MCD = 30°$. As $\sin 30° = \frac{1}{2}$, so $2MD = DC$. This means that $BD = MD$ and $\triangle MDB$ is isosceles. It follows that $\angle MBD = 30°$ and $\angle ABM = 15°$.

We further observe that $\triangle MBC$ is also isosceles and thus $MB = MC$.

Now $\angle BAM = \angle BMD - \angle ABM = 15°$, giving us yet another isosceles triangle $\triangle BAM$. We now have $MC = MB = MA$, so $\triangle AMC$ is also isosceles. This allows us to calculate $\angle ACM = 45°$ and finally $\angle ACB = 30° + 45° = 75°$.



32. Answer: 221

We have $a^2+2ab-3b^2 = (a-b)(a+3b) = 41$. Since 41 is a prime number, and $a-b < a+3b$, we have $a - b = 1$ and $a + 3b = 41$. Solving the simultaneous equations gives $a = 11$ and $b = 10$. Hence $a^2 + b^2 = 221$.

33. Answer: 62

We first note that for $1 \leq r < k$, $\lfloor \frac{r}{k} \rfloor = 0$ and $\lfloor \frac{k}{k} \rfloor = 1$. The total number of terms up to $\lfloor \frac{N}{N} \rfloor$ is given by $\frac{1}{2}N(N+1)$, and we have the inequality

$$\frac{62(63)}{2} = 1953 < 2013 < 2016 = \frac{63(64)}{2}.$$

So the 2013^{th} term is $\lfloor \frac{60}{63} \rfloor$, and the sum up to this term is just 62.

34. Answer: 1648

By the pigeonhole principle in any group of $365 \times 9 + 1 = 3286$ persons, there must be at least 10 persons who share the same birthday.

Hence solving $2n - 10 \geq 3286$ gives $n \geq 1648$. Thus the smallest possible n is 1648 since $2 \times 1647 - 10 = 3284 < 365 \times 9$, and it is possible for each of the 365 different birthdays to be shared by at most 9 persons.

35. Answer: 64

Let $d > 1$ be the highest common factor of $n - 11$ and $3n + 20$. Then $d \mid (n - 11)$ and $d \mid (3n + 20)$. Thus $d \mid [3n + 20 - 3(n - 11)]$, i.e., $d \mid 53$. Since 53 is a prime and $d > 1$, it follows that $d = 53$. Therefore $n - 11 = 53k$, where k is a positive integer, so $n = 53k + 11$. Note that for any k, $3n+20$ is a multiple of 53 since $3n+20 = 3(53k+11)+20 = 53(3k+1)$. Hence $n = 64$ (when $k = 1$) is the smallest positive integer such that $\text{HCF}(n-11, 3n+20) > 1$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2013

(Junior Section, Round 2)

Saturday, 29 June 2013 **0930-1230**

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1. Let $a < b < c < d < e$ be real numbers. Among the 10 sums of the pairs of these numbers, the least three are 32, 36 and 37 while the largest two are 48 and 51. Find all possible values of e.

2. In the triangle ABC, points D, E, F are on the sides BC, CA and AB respectively such that FE is parallel to BC and DF is parallel to CA. Let P be the intersection of BE and DF, and Q the intersection of FE and AD. Prove that PQ is parallel to AB.

3. Find all primes that can be written both as a sum of two primes and as a difference of two primes.

4. Let a and b be positive integers with $a > b > 2$. Prove that $\frac{2^a+1}{2^b-1}$ is not an integer.

5. Six musicians gathered at a chamber music festival. At each scheduled concert some of the musicians played while the others listened as members of the audience. What is the least number of such concerts which would need to be scheduled so that for every two musicians each must play for the other in some concert?

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(Junior Section, Round 2 solutions)

1. We have 37 is either $a+d$ or $b+c$ and

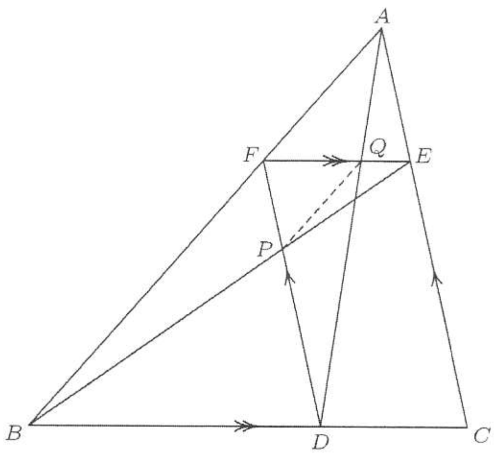
$$a+b=32, \quad a+c=36, \quad c+e=48, \quad d+e=51$$

Thus $c-b=4$, $d-c=3$ and $d-b=7$. Therefore $(a+b)+(d-b)=a+d=39$. Hence $b+c=37$. We thus have $a=15.5$, $b=16.5$, $c=20.5$, $d=23.5$ and $e=27.5$.

2. Since FE is parallel to BC and DF is parallel to CA, we have the triangles PFE, PDB and ECB are similar. Also the triangles AFQ and ABD are similar, FBD and ABC are similar. It follows that

$$\frac{DP}{PF}=\frac{BP}{PE}=\frac{BD}{DC}=\frac{BF}{FA}=\frac{DQ}{QA}$$

so that PQ is parallel to AB.



3. Let p be such a prime, then $p>2$ and is therefore odd. Thus $p=q-2=r+2$ where q,r are primes. If $r\equiv 1 \pmod 3$, then $p\equiv 0 \pmod 3$ and therefore $p=3$ and $r=1$ which is impossible. If $r\equiv 2 \pmod 3$, then $q\equiv 0 \pmod 3$ and thus $q=3$ and so $p=1$, again impossible. Thus $r\equiv 0 \pmod 3$, which means $r=3$ and hence $p=5$ and $q=7$. Thus $p=5$ is the only such prime.

4. We have $a = bm + r$ where $m = \lfloor a/b \rfloor$ and $0 \le r < b$. Thus

$$\frac{2^a+1}{2^b-1} = \frac{2^a-2^r}{2^b-1} + \frac{2^r+1}{2^b-1}.$$

Note that $2^a - 2^r = 2^r(2^{a-r}-1) = 2^r(2^{bm}-1)$, and

$$2^{bm} - 1 = (2^b)^m - 1 = (2^b-1)[(2^b)^{m-1} + (2^b)^{m-2} + \cdots + 1].$$

Therefore $\frac{2^a-2^r}{2^b-1}$ is an integer.

Observe that if $b > 2$, then $2^{b-1}(2-1) > 2$, i.e.,

$$2^r + 1 \le 2^{b-1} + 1 < 2^b - 1.$$

Therefore $\frac{2^r+1}{2^b-1}$ is not an integer. Thus $\frac{2^a+1}{2^b-1}$ is not an integer.

5. Let the musicians be A, B, C, D, E, F. We first show that four concerts are sufficient. The four concerts with the performing musicians: $\{A, B, C\}$, $\{A, D, E\}$, $\{B, D, F\}$ and $\{C, E, F\}$ satisfy the requirement. We shall now prove that 3 concerts are not sufficient. Suppose there are only three concerts. Since everyone must perform at least once, there is a concert where two of the musicians, say A, B, played. But they must also play for each other. Thus we have A played and B listened in the second concert and vice versa in the third. Now C, D, E, F must all perform in the second and third concerts since these are the only times when A and B are in the audience. It is not possible for them to perform for each other in the first concert. Thus the minimum is 4.