Multiple Choice Questions

1. A shop sells two kinds of products A and B. One day, a salesman sold both A and B at the same price \$2100 to a customer. Suppose product A makes a profit of 20% but product B makes a loss of 20%. Then this deal

(A) make a profit of \$70; (B) make a loss of \$70; (C) make a profit of \$175;
(D) make a loss of \$175; (E) makes no profit or loss.

D

2. How many integer solutions does the equation $(x^3 - x - 1)^{x+2013} = 1$ have?

(A) 0; (B) 1; (C) 2; (D) 3; (E) More than 3.

D

3. In the xy-plane, which of the following is the reflection of the graph of

$$y = \frac{1+x}{1+x^2}$$

about the line $y = 2x$?

(A) $x = \frac{1+y}{1+y^2}$; (B) $x = \frac{-1+y}{1+y^2}$; (C) $x = -\frac{1+y}{1+y^2}$; (D) $x = \frac{1-y}{1+y^2}$;
(E) None of the above.

E

4. Let n be a positive integer. Find the number of possible reminders when

$$2013^n - 1803^n - 1781^n + 1774^n$$

is divided by 203.

A

(A) 1; (B) 2; (C) 3; (D) 4; (E) More than 4.

5. Find the number of integers n such that the equation

$$xy^2 + y^2 - x - y = n$$

has an infinite number of integer solutions (x, y).

(A) 0; (B) 1; (C) 2; (D) 3; (E) More than 3.

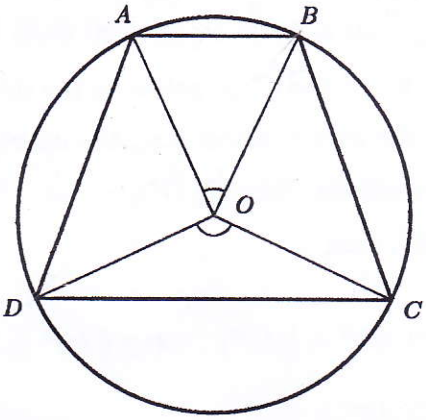
C

6. If $0 < \theta < \frac{\pi}{4}$ is such that $\operatorname{cosec}\theta - \sec\theta = \frac{\sqrt{13}}{6}$, then $\cot\theta - \tan\theta$ equals

(A) $\frac{\sqrt{13}}{6}$; (B) $\frac{\sqrt{12}}{6}$; (C) $\frac{\sqrt{5}}{6}$; (D) $\frac{13}{6}$; (E) $\frac{5}{6}$.

E

7. $ABCD$ is a trapezium inscribed in a circle centered at O. It is given that $AB \parallel CD$, $\angle COD = 3\angle AOB$, and $\dfrac{AB}{CD} = \dfrac{2}{5}$.

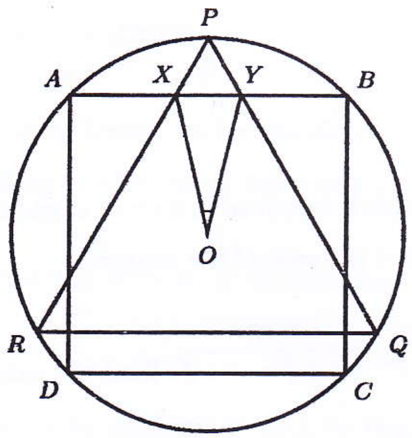


Find the ratio

$$\frac{\text{area of } \triangle BOC}{\text{area of } \triangle AOB}.$$

(A) $\frac{3}{2}$; (B) $\frac{7}{4}$; (C) $\frac{\sqrt{3}}{\sqrt{2}}$; (D) $\frac{\sqrt{5}}{2}$; (E) $\frac{\sqrt{7}}{\sqrt{2}}$. A

8. A square $ABCD$ and an equilateral triangle PQR are inscribed in a circle centered at O in such a way that $AB \parallel QR$. The sides PQ and PR of the triangle meet the side AB of the square at X and Y respectively.



The value of $\tan \angle XOY$ is

(A) $\frac{1}{\sqrt{3}}$; (B) 1; (C) $\frac{\sqrt{6}-\sqrt{3}}{\sqrt{2}}$; (D) $\frac{2\sqrt{2}-2}{\sqrt{3}}$; (E) $\sqrt{3}$. C

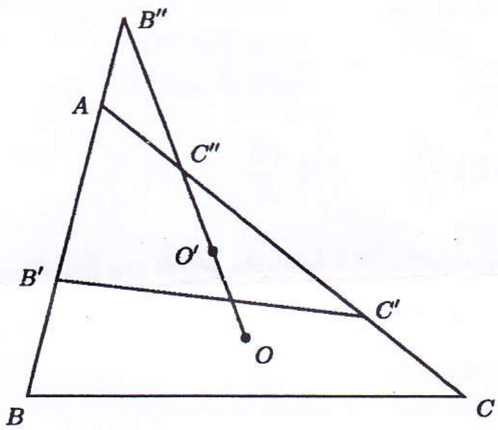
9. Two people go to the same swimming pool between 2:00p.m. and 5:00p.m. at random time and each swims for one hour. What is the chance that they meet?

(A) $\frac{1}{9}$; (B) $\frac{2}{9}$; (C) $\frac{1}{3}$; (D) $\frac{4}{9}$; (E) $\frac{5}{9}$. E

10. Given a triangle $\triangle ABC$, let B' and C' be points on the sides AB and AC such that $BB' = CC'$. Let O and O' be the circumcentres (i.e., the centre of the circumscribed circle) of $\triangle ABC$ and $\triangle AB'C'$, respectively. Suppose OO' intersect lines AB' and AC' at B'' and C'', respectively. If $AB = \frac{1}{2}AC$, then

(A) $AB'' < \frac{1}{2}AC''$; (B) $AB'' = \frac{1}{2}AC''$; (C) $\frac{1}{2}AC'' < AB'' < AC''$;

(D) $AB'' = AC''$; (E) $AB'' > AC''$.



D

Short Questions

11. Suppose a right-angled triangle is inscribed in a circle of radius 100. Let α and β be its acute angles. If $\tan\alpha = 4\tan\beta$, find the area of the triangle. 8000

12. Let $f(x) = \dfrac{1+10x}{10-100x}$. Set $f^n = \overbrace{f \circ f \circ \cdots \circ f}^{n \text{ terms}}$. Find the value of

$$f\left(\frac{1}{2}\right) + f^2\left(\frac{1}{2}\right) + f^3\left(\frac{1}{2}\right) + \cdots + f^{6000}\left(\frac{1}{2}\right).$$

595

13. Let AB and CD be perpendicular segments intersecting at point P. Suppose that $AP = 2$, $BP = 3$ and $CP = 1$. If all the points A, B, C, D lie on a circle, find the length of DP.

6

14. On the xy-plane, let S denote the region consisting of all points (x, y) for which

$$\left|x+\frac{1}{2}y\right| \leq 10, \quad |x| \leq 10 \quad \text{and} \quad |y| \leq 10.$$

The largest circle centred at $(0,0)$ that can be fitted in the region S has area $k\pi$. Find the value of k. 80

15. Given that $\sqrt[3]{17-\frac{27}{4}\sqrt{6}}$ and $\sqrt[3]{17+\frac{27}{4}\sqrt{6}}$ are the roots of the equation

$$x^2 - ax + b = 0,$$

find the value of ab. 10

16. Find the number of integers between 1 and 2013 with the property that the sum of its digits equals 9. 101

17. Let $p(x)$ be a polynomial with integer coefficients such that $p(m) - p(n)$ divides $m^2 - n^2$ for all integers m and n. If $p(0) = 1$ and $p(1) = 2$, find the largest possible value of $p(100)$. 10001

18. Find the number of positive integer pairs (a, b) satisfying $a^2 + b^2 < 2013$ and $a^2b \mid (b^3 - a^3)$. 31

19. Let f and g be functions such that for all real numbers x and y,

$$g(f(x+y)) = f(x) + (x+y)g(y).$$

Find the value of $g(0) + g(1) + \cdots + g(2013)$. 0

20. Each chocolate costs 1 dollar, each licorice stick costs 50 cents and each lolly costs 40 cents. How many different combinations of these three items cost a total of 10 dollars? 36

21. Let $A = \{1, 2, 3, 4, 5, 6\}$. Find the number of distinct functions $f : A \to A$ such that $f(f(f(n))) = n$ for all $n \in A$. 81

22. Find the number of triangles whose sides are formed by the sides and the diagonals of a regular heptagon (7-sided polygon). (Note: The vertices of triangles need not be the vertices of the heptagon.) ~~168~~ ~~1430~~ ~~357~~ 287

23. Six seats are arranged in a circular table. Each seat is to be painted in red, blue or green such that any two adjacent seats have different colours. How many ways are there to paint the seats?

66

24. $\triangle ABC$ is an equilateral triangle of side length 30. Fold the triangle so that A touches a point X on BC. If $BX = 6$, find the value of k, where $\sqrt{k}$ is the length of the crease obtained from folding.

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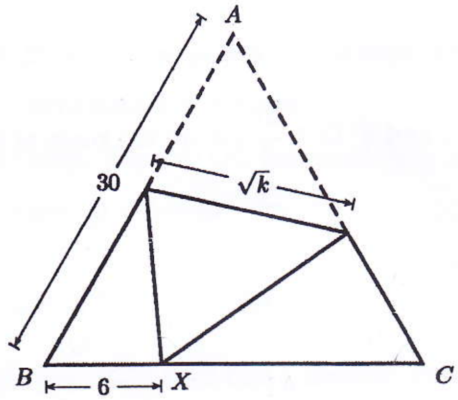

25. As shown in the figure below, circles C_1 and C_2 of radius 360 are tangent to each other, and both tangent to straight line ℓ. If circle C_3 is tangent to C_1, C_2 and ℓ, and circle C_4 is tangent to C_1, C_3 and ℓ, find the radius of C_4.

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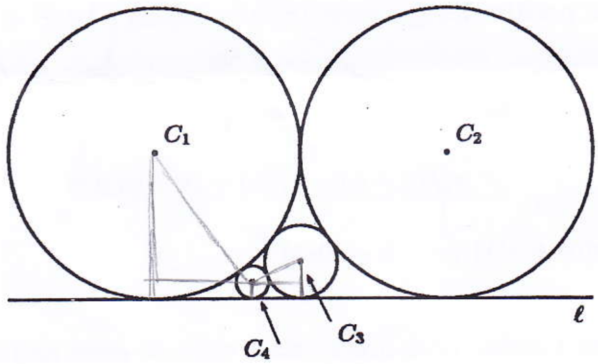


26. Set $\{x\} = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. Find the number of real solutions to the equation

181

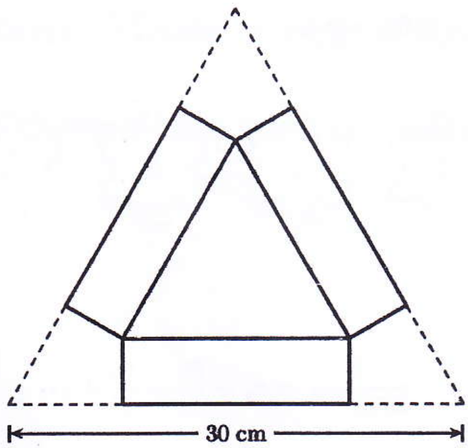
$$\{x\} + \{x^2\} = 1, \quad |x| \leq 10.$$

27. Find the value of $\left\lfloor \left(\frac{3+\sqrt{17}}{2} \right)^6 \right\rfloor$. 2040

28. A regular dodecagon (12-sided polygon) is inscribed in a circle of radius 10. Find its area.

300

29. A triangular box is to be cut from an equilateral triangle of length 30 cm. Find the largest possible volume of the box (in cm).



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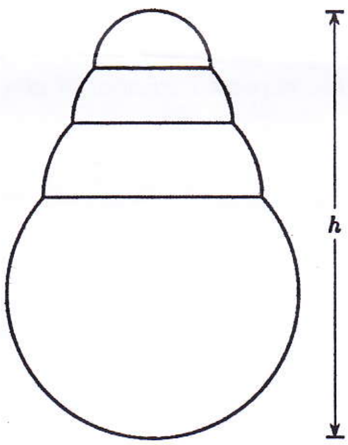
30. A hemisphere is placed on a sphere of radius 100 cm. The second hemisphere is placed on the first one, and the third hemisphere is placed on the second one (as shown below). Find the maximum height of the tower (in cm).



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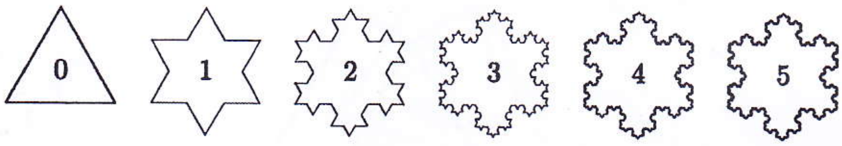
31. Let x, y, z be real numbers such that

$$x + y + z = 1 \quad \text{and} \quad x^2 + y^2 + z^2 = 1.$$

5

Let m denote the minimum value of $x^3 + y^3 + z^3$. Find $9m$.

32. Given an equilateral triangle of side 10, divide each side into three equal parts, construct an equilateral triangle on the middle part, and then delete the middle part. Repeat this step for each side of the resulting polygon. Find S^2, where S is the area of region obtained by repeating this procedure infinitely many times.



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33. Suppose

$$\frac{1}{2013^{1000}} = a_1 + \frac{a_2}{2!} + \frac{a_3}{3!} + \cdots + \frac{a_n}{n!},$$

where n is a positive integer, $a_1, \ldots, a_n$ are nonnegative integers such that $a_k < k$ for $k = 2, \ldots, n$ and $a_n > 0$. Find the value of n.

60024

34. Let M be a positive integer. It is known that whenever $|ax^2 + bx + c| \leq 1$ for all $|x| \leq 1$, then $|2ax + b| \leq M$ for all $|x| \leq 1$. Find the smallest possible value of M.

4

35. Consider integers $\{1, 2, \ldots, 10\}$. A particle is initially at 1. It moves to an adjacent integer in the next step. What is the expected number of steps it will take to reach 10 for the first time?

81